Die Wiederholungsaufgaben zur letzten Vorlesung sind Online.

Bitte bearbeiten!

Integration durch einfache Subst.

Da es für ein Integral der Form

$$\int \frac{1}{x+2}\,dx$$

keine eigene Integrationsregel gibt, wendet man einen Trick an. Man ersetzt den durch die Summe kompliziert gewordenen Nenner durch eine neue Variable u (Substitution), somit entsteht ein berechenbares Interval.

Beispiel: $\int \frac{1}{x+2}\,dx$

$$u = x+2$$

Diesen Variablenwechsel von der Variablen x zu einer neuen Variablen u muss man konsequent weiterführen, d.h. man muss dx auf u umrechnen.
Wir müssen uns erinnern, dass dx ein Differential ist
Für Differentiale gibt es diese Formel:

$$\boxed{dy = y'dx \quad \text{oder} \quad du = u' \cdot dx}$$

Beispiel: $\int \frac{1}{x+2}\,dx$

Subst: $u = x+2$

$$\int \frac{1}{u}\,dx$$

Dazu bilden wir das Differential

$$\left.\begin{array}{l} du = u'dx \\ u' = 1 \end{array}\right\} \Rightarrow \quad \begin{array}{l} du = 1 \cdot dx \\ \boxed{du = dx} \end{array}$$

$$\Rightarrow \int \frac{1}{x+2}\,dx = \int \frac{1}{u}\,du = \ln|u| + c$$

Nach der Berechnung muss wieder die Rücksubst. stattfinden.

$$= \underline{\underline{\ln|x+2| + c}}$$

$$\int \frac{1}{(x+2)^2}\,dx$$

Subst: $(x+2)^2 = u$

$u' = 2\cdot(x+2)$

$du = u'\cdot dx$

$du = 2(x+2)\,dx$

⇒ Wichtig auch zu erkennen, ob der Weg zu einem einfacheren Integral führt.

Subst: $u = x+2$

$u' = 1$

$du = u'\cdot dx$

$du = 1\cdot dx \Rightarrow du = dx$

$$\int \frac{1}{(x+2)^2}\,dx = \int \frac{1}{u^2}\cdot du = \int u^{-2}\,du$$

Rücksub.

$$= \frac{u^{-1}}{-1} + c = -\frac{1}{u} + c = \underline{\underline{-\frac{1}{x+2} + c}}$$

$\int \frac{1}{(4x-1)^2} dx$

$u = 4x-1$

$u' = 4$

$du = u' \cdot dx \quad \Rightarrow \quad du = 4 \cdot dx \quad | :4$

$\frac{1}{4} \cdot du = dx$

$\int \frac{1}{u^2} \cdot \frac{1}{4} du = \frac{1}{4} \int u^{-2} du$

$= \frac{1}{4} \cdot \frac{u^{-1}}{-1} + c = -\frac{1}{4} \cdot \frac{1}{u} + c$

Rücksubst.: $\underline{\underline{-\frac{1}{4 \cdot (4x-1)} + c}}$

$$\int \frac{5}{2x-1}\,dx = 5 \cdot \int \frac{1}{2x-1}\,dx$$

$u = 2x-1$

$u' = 2$

$du = u' \cdot dx$

$du = 2 \cdot dx \quad \Rightarrow \quad \frac{1}{2} du = dx$

$$5 \cdot \int \frac{1}{u} \cdot \frac{1}{2}\,du = \frac{5}{2} \int \frac{1}{u}\,du$$

$$= \frac{5}{2} \cdot \ln|u| + c = \underline{\frac{5}{2} \cdot \ln|2x+1| + c}$$

Aufgaben 53

(16) $\int \frac{4}{(4x+5)^2}\,dx$

$u = 4x+5$

$u' = 4 \quad \Rightarrow \quad du = 4\,dx \Rightarrow \frac{1}{4}du = dx$

$= 4\int \frac{1}{u^2} \cdot \frac{1}{4}du = \int \frac{1}{u^2}du = \int u^{-2}du$

$= \frac{u^{-1}}{-1} + C = -\frac{1}{u} + C = -\frac{1}{4x+5} + C$

(17) $\int \frac{2}{(1-x)^3}\,dx$

$u = 1-x$

$u' = -1 \Rightarrow du = -1 \cdot dx \Rightarrow -du = dx$

$= 2 \cdot \int \frac{1}{u^3} - du = -2\int u^{-3}du$

$= -2 \cdot \frac{u^{-2}}{-2} + C = u^{-2} + C = \frac{1}{u^2} + C = \underline{\frac{1}{(1-x)^2} + C}$

(18) $\int \frac{24}{(x+12)^2}\,dx$

$u = x+12$

$u' = 1 \Rightarrow du = 1 \cdot dx \Rightarrow du = dx$

$= 24 \int \frac{1}{u^2}\,du = 24 \int u^{-2}\,du = 24 \cdot \frac{u^{-1}}{-1} + c$

$= -24 \cdot \frac{1}{u} + c = \underline{-\frac{24}{x+12} + c}$

(19) $\int \frac{3}{3-x}\,dx$

$u = 3-x$

$u' = -1 \Rightarrow du = -1 \cdot dx \Rightarrow -du = dx$

$= 3 \cdot \int \frac{1}{u} \cdot -du = -3 \int \frac{1}{u}\,du$

$= -3 \cdot \ln|u| + c = \underline{-3 \cdot \ln|3-x| + c}$

(20) $\int \frac{-3}{(2x-4)^4}\,dx$

$u = 2x-4$

$u' = 2 \Rightarrow du = 2dx \Rightarrow \frac{1}{2}du = dx$

$= -3\int \frac{1}{u^4}\cdot\frac{1}{2}\,du = -\frac{3}{2}\int\frac{1}{u^4}\,du = -\frac{3}{2}\int u^{-4}\,du$

$= -\frac{3}{2}\cdot\frac{u^{-3}}{-3} + C = \frac{1}{2}\cdot\frac{1}{u^3} + C = \underline{\frac{1}{2\cdot(2x-4)^3}} + C$

(21) $\int \frac{12}{(x-2)^5}\,dx$

$u = x-2$

$u' = 1 \Rightarrow du = dx$

$= 12\int\frac{1}{u^5}\,du = 12\cdot\int u^{-5}\,du = 12\cdot\frac{u^{-4}}{-4} + C$

$= -3\cdot\frac{1}{u^4} + C = \underline{-\frac{3}{(x-2)^4} + C}$

Der Nenner enthält eine Summe und der Zähler auch die Var. x

Die erweiterte Substitution

Dieses x im Zähler erschwert die Rechnung.

Beispiel: $\int \frac{x}{(5x+2)^2}\,dx$

1. Schritt: Subst. $u = 5x+2$

2. Schritt: Differential: $du = u' \cdot dx$

$u' = 5$

$du = 5dx$

3. Schritt: Nach dx auflösen: $\frac{1}{5}du = dx$

$$\int \frac{x}{u^2} \cdot \frac{1}{5} \cdot du$$

4. Schritt: Subst. wird nach x aufgelöst und eingesetzt.

$u = 5x+2$

$5x = u-2$

$x = \frac{1}{5}(u-2)$

5. Schritt: Einsetzen und berechnen

$$\int \frac{x}{u^2} \cdot \frac{1}{5} du = \int \frac{\frac{1}{5}(u-2)}{u^2} \cdot \frac{1}{5} du$$

$$= \frac{1}{25} \int \frac{(u-2)}{u^2} du = \frac{1}{25} \int \left(\frac{u}{u^2} - \frac{2}{u^2}\right) du$$

$$= \frac{1}{25} \int \left(\frac{1}{u} - \frac{2}{u^2}\right) du = \frac{1}{25} \cdot \left(\ln|u| + \frac{2}{u}\right) + C$$

$$= \frac{1}{25} \cdot \left(\ln|5x+2| + \frac{2}{5x+2}\right) + C$$

Beispiel: $\int \frac{x^2}{(x+3)^2}\,dx$

$u = x+3$; $u' = 1$; $du = dx$; $x = u-3$

$$\int \frac{(u-3)^2}{u^2}\,du = \int \frac{u^2-6u+9}{u^2}\,du$$

$$= \int \left(\frac{u^2}{u^2} - \frac{6u}{u^2} + \frac{9}{u^2}\right) du = \int \left(1 - \frac{6}{u} + \frac{9}{u^2}\right) du$$

$$= u - 6 \cdot \ln|u| - \frac{9}{u} + C$$

$$= x+3 - 6 \cdot \ln|x+3| - \frac{9}{x+3} + C$$

$$\int \frac{x^2}{(2-x)^2}\, dx$$

$u = 2-x$; $u' = -1$; $-du = dx$; $x = 2-u$

$$= \int \frac{(2-u)^2}{u^2} \cdot -du = -\int \frac{(2-u)^2}{u^2}\, du$$

$$= -\int \frac{4-4u+u^2}{u^2}\, du = -\int \left(\frac{4}{u^2} - \frac{4}{u} + 1\right) du$$

$$= +\frac{4}{u} + 4 \cdot \ln|u| - u + C$$

$$= +\frac{4}{2-x} + 4 \cdot \ln|2-x| - (2-x) + C$$

$$= +\frac{4}{2-x} + 4 \cdot \ln|2-x| + x - 2 + C$$

Beispiel: $\int \frac{x^2-1}{x+2}\,dx$

$u = x+2$; $u' = 1$; $du = dx$; $x = u-2$

$= \int \frac{(u-2)^2-1}{u}\,du = \int \frac{u^2-4u+4-1}{u}\,du$

$= \int \frac{u^2-4u+3}{u}\,du = \int \left(\frac{u^2}{u} - \frac{4u}{u} + \frac{3}{u}\right) du$

$= \int \left(u - 4 + \frac{3}{u}\right) du = \frac{1}{2}u^2 - 4u + 3 \cdot \ln|u| + c$

$= \frac{1}{2}(x+2)^2 - 4(x+2) + 3 \cdot \ln|x+2| + c$

$= \frac{1}{2}x^2 + 2x + 2 - 4x - 8 + 3 \cdot \ln|x+2| + c$

$= \frac{1}{2}x^2 - 2x - 6 + 3 \cdot \ln|x+2| + c$

Zu Übung rechnen Sie
bitte folgende Aufgaben
aus Ihrer Aufgabensammlung

54/55

(22) $\int \frac{2x}{(x+1)^2}\,dx$

$u = x+1$; $u' = 1$; $du = dx$; $x = u-1$

$= \int \frac{2(u-1)}{u^2}\,du = \int \frac{2u-2}{u^2}\,du = \int \left(\frac{2u}{u^2} - \frac{2}{u^2}\right) du$

$= \int \left(\frac{2}{u} - \frac{2}{u^2}\right) du = 2 \cdot \ln|u| + \frac{2}{u} + c$

$= 2 \cdot \ln|x+1| + \frac{2}{x+1} + c$

(23) $\int \frac{x^2}{(4-x)^2}\,dx$

$u = 4-x$; $u' = -1$; $-du = dx$; $x = 4-u$

$= \int \frac{(4-u)^2}{u^2} \cdot -du = -\int \frac{16-8u+u^2}{u^2}\,du$

$= -\int \left(\frac{16}{u^2} - \frac{8}{u} + 1\right) du = + \frac{16}{u} + 8 \cdot \ln|u| - u + c$

$= \frac{16}{4-x} + 8 \cdot \ln|4-x| - 4 + x + C$

(24) $\int \frac{x^2+x-1}{(2x+4)^2}\,dx$

$u=2x+4$; $u'=2$; $\frac{1}{2}du=dx$; $x=\frac{1}{2}(u-4)$

$= \int \frac{\frac{1}{4}(u-4)^2+\frac{1}{2}(u-4)-1}{u^2}\cdot\frac{1}{2}du$

$= \frac{1}{2}\int \frac{\frac{1}{4}u^2-2u+4+\frac{1}{2}u-2-1}{u^2}\,du$

$= \frac{1}{2}\int \frac{\frac{1}{4}u^2-\frac{3}{2}u+1}{u^2}\,du = \frac{1}{2}\int\left(\frac{1}{4}-\frac{3}{2}\cdot\frac{1}{u}+\frac{1}{u^2}\right)du$

$= \frac{1}{2}\left(\frac{1}{4}u-\frac{3}{2}\cdot\ln|u|-\frac{1}{u}\right)+c$

$= \frac{1}{8}u-\frac{3}{4}\cdot\ln|u|-\frac{1}{2u}+c$

$= \frac{1}{8}\cdot(2x+4)-\frac{3}{4}\cdot\ln|2x+4)-\frac{1}{2(2x+4)}+c$

$= \frac{1}{4}(x+2)-\frac{3}{4}\ln|2x+4|-\frac{1}{4x+8}+c$

(25) $\int \frac{x}{x-1}\,dx$

$u = x-1 \,;\; u' = 1 \,;\; du = dx \,;\; x = u+1$

$\int \frac{u+1}{u}\,du = \int \left(\frac{u}{u} + \frac{1}{u}\right) du = \int \left(1 + \frac{1}{u}\right) du$

$= u + \ln|u| + C = \underline{x - 1 + \ln|x-1| + C}$

(26) $\int \frac{2-x}{4x+1}\,dx$

$u = 4x+1 \;;\; u' = 4 \;;\; \frac{1}{4}du = dx \;;\; x = \frac{1}{4}(u-1)$

$= \int \frac{2-\frac{1}{4}(u-1)}{u}\,\frac{1}{4}du = \frac{1}{4}\int \frac{2-\frac{1}{4}u+\frac{1}{4}}{u}\,du$

$= \frac{1}{4}\int \frac{\frac{9}{4}-\frac{1}{4}u}{u}\,du = \frac{1}{4}\int\left(\frac{9}{4}\cdot\frac{1}{u}-\frac{1}{4}\right)du$

$= \frac{1}{4}\left(\frac{9}{4}\cdot\ln|u| - \frac{1}{4}u\right) + c$

$= \frac{9}{16}\cdot\ln|u| - \frac{1}{16}u + c$

$= \frac{9}{16}\cdot\ln|4x+1| - \frac{1}{16}(4x+1) + c$

(27) $\int \frac{(x+2)^2}{2-x}\,dx$

$u = 2-x$; $u' = -1$; $-du = dx$; $x = 2-u$

$= \int \frac{(2-u+2)^2}{u} \cdot -du = -\int \frac{(4-u)^2}{u}\,du$

$= -\int \frac{16-8u+u^2}{u}\,du = -\int \left(\frac{16}{u} - 8 + u\right) du$

$= -16 \cdot \ln|u| + 8u - \frac{1}{2}u^2 + C$

$= -16 \cdot \ln|2-x| + 8(2-x) - \frac{1}{2}(2-x)^2 + C$

$= -16 \cdot \ln|2-x| + 16 - 8x - 2 + 2x - \frac{1}{2}x^2 + C$

$= -16 \cdot \ln|2-x| + 14 - 6x - \frac{1}{2}x^2 + C$

$$\int \frac{16x}{x^2+12}\,dx$$

$u = x^2+12 \; ; \; u' = 2x \; ; \; du = u' \cdot dx$

$du = 2x \cdot dx$

$\frac{1}{2} du = x \cdot dx$

$$= 16 \int \frac{x}{x^2+12}\,dx = 16 \int \frac{1}{u} \cdot \frac{1}{2}\,du$$

$$= 8 \int \frac{1}{u}\,du = 8 \cdot \ln|u| + C$$

$$= 8 \cdot \ln|x^2+12| + C$$

$$\int \frac{2x}{(x^2-4)^2} dx$$

$$u = x^2-4; \quad u' = 2x; \quad du = u' \cdot dx$$

$$du = 2x\, dx$$

$$\frac{1}{2} du = x\, dx$$

$$= 2 \cdot \int \frac{1}{u^2} \cdot \frac{1}{2} du = \int \frac{1}{u^2} du$$

$$= \left(-\frac{1}{u}\right) + C = -\frac{1}{x^2-4} + C$$

$$\int \frac{x+1}{x^2+2x}\,dx$$

$u = x^2+2x \;;\; u' = 2x+2 \Rightarrow u' = 2(x+1)$

$du = u' \cdot dx$

$du = 2(x+1)\,dx$

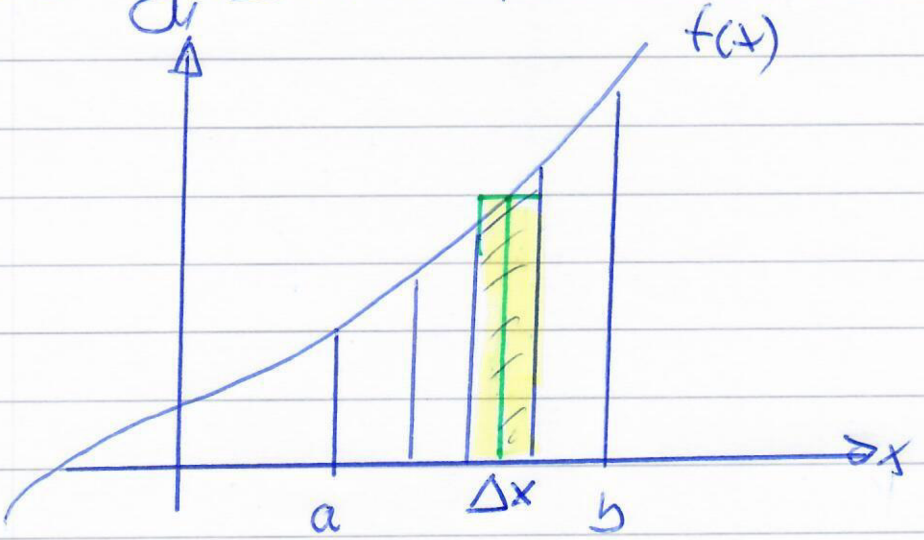
$\frac{1}{2}du = (x+1)\,dx$

$$= \int \frac{1}{u} \cdot \frac{1}{2}\,du = \frac{1}{2}\int \frac{1}{u}\,du$$

$$= \frac{1}{2}\ln|u| + C = \frac{1}{2} \cdot \ln|x^2+2x| + C$$

Das bestimmte Integral

Die Funktion f(x) sei gegeben;
wir wollen die Fläche zwischen
dem Funktionsgraphen und der
x-Achse im Intervall [a;b]
berechnen.
Einen Näherungswert erhält man,
wenn man [a;b] in Teilintervalle
der Länge Δx teilt,



in jedem Intervall die Mitte x_i wählt
und die Flächen der einzelnen
Rechtecke addiert

Die Fläche - das bestimmte Integral - definieren wir als Grenzwert dieser Summe, wenn Δx gegen Null geht, so schreibt man:

$$\boxed{A = \int_a^b f(x)\,dx}$$

Hauptsatz der Integralrechnung

$$\boxed{\int_a^b f(x)\,dx = F(b) - F(a)}$$

Beispiel: Wir suchen die Fläche unter dem Graphen der Funktion $f(x) = x^2$ zwischen den Grenzen $a = 1$ und $b = 2$.

$$\int_1^2 x^2\,dx = \left[\frac{1}{3}x^3 + \cancel{C}\right]_1^2$$

$$= \frac{1}{3}\cdot 2^3 + \cancel{C} - \frac{1}{3}1^3 - \cancel{C} = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}\ [FE]$$

Gesetzmäßigkeiten

$$\boxed{\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx}$$

Beispiel:

$$\int_3^1 x^2\,dx = \left[\frac{1}{3}x^3\right]_3^1 = \frac{1}{3} - 9 = -\frac{26}{3}$$

$$\int_1^3 x^2\,dx = \left[\frac{1}{3}x^3\right]_1^3 = 9 - \frac{1}{3} = +\frac{26}{3}$$

$$\boxed{\int_a^a f(x)\,dx = 0}$$

$$\boxed{\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx \quad \text{mit } a < b < c}$$

Lösen Sie folgende Aufgaben aus ihrer Aufgabensammlung

56-3 56-17 56-22
57-2 57-9
58-13 58-18
59-5 59-7 59-8
60-10 60-13
61
63-8
64-11
66-17 66-19
67-22 67-24

$$\int_4^6 \frac{5}{2x+4}\,dx = 5\int_4^6 \frac{1}{2x+4}\,dx$$

$u = 2x+4$; $u' = 2$; $\frac{du}{dx} = 2$; $\frac{1}{2}du = dx$

$$= 5\int_4^6 \frac{1}{u}\cdot\frac{1}{2}\,du = \frac{5}{2}\cdot\int_{x_1=4}^{x_2=6} \frac{1}{u}\,du$$

$$= \frac{5}{2}\cdot\Big[\ln|u|\Big]_{4=x_1}^{6=x_2}$$

Rücksubst. bevor die Grenzen eingesetzt werden.

$$= \frac{5}{2}\cdot\Big[\ln|2x+4|\Big]_4^6$$

$$= \frac{5}{2}\cdot\ln|2\cdot 6+4| - \frac{5}{2}\cdot\ln|2\cdot 4+4|$$

$$= \frac{5}{2}\Big[\ln|16| - \ln|12|\Big] = \frac{5}{2}\cdot\ln\left|\frac{16}{12}\right|$$

$$= \frac{5}{2}\cdot\ln\left|\frac{4}{3}\right|$$

$$\int_4^6 \frac{5}{2x+4}\,dx = 5\int_4^6 \frac{1}{2x+4}\,dx$$

$u = 2x+4$; $u' = 2$; $\frac{du}{dx} = 2$; $\frac{1}{2}du = dx$

Die Grenzen werden auch nach u subst. Dann muss am Schluß nicht rücksubst. werden, sondern die Grenzen können direkt eingesetzt werden.

$x_1 = 4 \Rightarrow u_1 = 2\cdot 4+4 = 12$

$x_2 = 6 \Rightarrow u_2 = 2\cdot 6+4 = 16$

$$= 5\int_{12}^{16} \frac{1}{u}\cdot\frac{1}{2}\,du = \frac{5}{2}\int_{12}^{16} \frac{1}{u}\,du$$

$$= \frac{5}{2}\cdot\left[\ln|u|\right]_{12}^{16} = \frac{5}{2}\cdot\left[\ln|16| - \ln|12|\right]$$

$$= \frac{5}{2}\cdot\ln\left|\frac{4}{3}\right|$$

$$\int_{-1}^{0} \frac{2-x}{(2x-5)^2}\,dx$$

$u = 2x-5$; $u' = 2$; $\frac{du}{dx} = 2$; $\frac{1}{2}du = dx$

$x = \frac{1}{2}(u+5)$

$x_1 = -1 \Rightarrow u_1 = 2\cdot(-1)-5 = -7$

$x_2 = 0 \Rightarrow u_2 = 2\cdot 0 - 5 = -5$

$$\int_{-7}^{-5} \frac{2-\frac{1}{2}(u+5)}{u^2}\cdot\frac{1}{2}\,du$$

$$= \frac{1}{2}\int_{-7}^{-5} \frac{2-\frac{1}{2}u-\frac{5}{2}}{u^2}\,du = \frac{1}{2}\int_{-7}^{-5} \frac{-\frac{1}{2}u-\frac{1}{2}}{u^2}\,du$$

$$= -\frac{1}{4}\int_{-7}^{-5} \frac{u+1}{u^2}\,du = -\frac{1}{4}\int_{-7}^{-5}\left(\frac{u}{u^2}+\frac{1}{u^2}\right)du$$

$$= -\frac{1}{4}\int_{-7}^{-5}\left(\frac{1}{u}+\frac{1}{u^2}\right)du = -\frac{1}{4}\left[\ln|u| - \frac{1}{u}\right]_{-7}^{-5}$$

$$= -\frac{1}{4}\left[\ln|-5| - \frac{1}{-5} - \ln|-7| + \frac{1}{-7}\right]$$

$$= -\frac{1}{4}\left[\ln|5| + \frac{1}{5} - \ln|7| - \frac{1}{7}\right.$$

$$= -\frac{1}{4}\left[\ln\left|\frac{5}{7}\right| + \frac{7-5}{35}\right] = -\frac{1}{4}\left[\ln\left|\frac{5}{7}\right| + \frac{2}{35}\right]$$

$$\int_2^3 \frac{2-x}{4x-x^2}\,dx$$

$u = 4x - x^2$; $u' = 4 - 2x = 2(2-x)$

$\frac{du}{dx} = 2(2-x)$; $\frac{1}{2}du = (2-x)dx$

$x_1 = 2$; $u_1 = 4 \cdot 2 - 2^2 = 4$

$x_2 = 3$; $u_2 = 4 \cdot 3 - 3^2 = 3$

$$= \int_4^3 \frac{1}{u} \cdot \frac{1}{2}du = \frac{1}{2}\int_4^3 \frac{1}{u}du = \left[\frac{1}{2}\ln|u|\right]_4^3$$

$$= \frac{1}{2}\left(\ln|3| - \ln|4|\right) = \frac{1}{2} \cdot \ln\left|\frac{3}{4}\right|$$

$$\int_1^4 \frac{4-x}{\sqrt{x}}\,dx = \int_1^4 \left(\frac{4}{\sqrt{x}} - \frac{x}{\sqrt{x}}\right) dx$$

$$= \int_1^4 \left(4 \cdot x^{-1/2} - x^{1/2}\right) dx = \left[\frac{4 \cdot x^{1/2}}{1/2} - \frac{x^{3/2}}{3/2}\right]_1^4$$

$$= \left[8 \cdot \sqrt{x} - \frac{2}{3} x \cdot \sqrt{x}\right]_1^4$$

$$= 8 \cdot \sqrt{4} - \frac{2}{3} \cdot 4\sqrt{4} - 8\sqrt{1} + \frac{2}{3} \cdot 1 \cdot \sqrt{1}$$

$$= 16 - \frac{16}{3} - 8 + \frac{2}{3} = 8 - \frac{14}{3} = \frac{24}{3} - \frac{14}{3} = \frac{10}{3}$$

$$\int_0^{4t}\left(x\cdot\sqrt{\frac{x}{t}}\right)dx = \int_0^{4t}\left(x\cdot\frac{\sqrt{x}}{\sqrt{t}}\right)dx$$

$$= \frac{1}{\sqrt{t}}\int_0^{4t} x^{3/2}\,dx = \left[\frac{1}{\sqrt{t}}\cdot\frac{x^{5/2}}{5/2}\right]_0^{4t}$$

$$= \left[\frac{2}{5\sqrt{t}}\cdot x^2\cdot\sqrt{x}\right]_0^{4t} = \frac{2}{5\sqrt{t}}\cdot 16t^2\cdot\sqrt{4t}-0$$

$$= \frac{32\,t^2\cdot 2\cancel{\sqrt{t}}}{5\,\cancel{\sqrt{t}}} = \frac{64}{5}t^2$$

$$\int_1^4 \frac{2x+1}{\sqrt{5-x}}\,dx$$

$u = 5-x \; ; \; u' = -1 \; ; \; \frac{du}{dx} = -1 \; ; \; -du = dx$

$x = 5-u \quad ; \quad x_1 = 1 \Rightarrow u_1 = 5-1 = 4$

$\qquad\qquad\qquad x_2 = 4 \Rightarrow u_2 = 5-4 = 1$

$$= \int_4^1 \frac{2(5-u)+1}{\sqrt{u}} \cdot -du = \int_1^4 \frac{10-2u+1}{\sqrt{u}}\,du$$

$$= \int_1^4 \frac{11-2u}{\sqrt{u}}\,du = \int_1^4 \left(\frac{11}{\sqrt{u}} - \frac{2u}{\sqrt{u}}\right) du$$

$$= \int_1^4 \left(11u^{-1/2} - 2u^{1/2}\right) du = \left[\frac{11 \cdot u^{1/2}}{1/2} - \frac{2 \cdot u^{3/2}}{3/2}\right]_1^4$$

$$= \left[22\sqrt{u} - \frac{4}{3}u \cdot \sqrt{u}\right]_1^4$$

$$= 22\sqrt{4} - \frac{4}{3} \cdot 4 \cdot \sqrt{4} - 22\sqrt{1} + \frac{4}{3} \cdot 1\sqrt{1}$$
$$= 44 - 32/3 - 22 + 4/3 = 38/3$$

$$\int_{\sqrt{5}}^{\sqrt{8}} \frac{x}{\sqrt{x^2-4}}\,dx$$

$u = x^2 - 4$; $u' = 2x$; $\frac{du}{dx} = 2x$; $\frac{1}{2}du = x\,dx$

$x_1 = \sqrt{5} \Rightarrow u_1 = \sqrt{5}^2 - 4 = 1$

$x_2 = \sqrt{8} \Rightarrow u_2 = \sqrt{8}^2 - 4 = 4$

$$\int_1^4 \frac{1}{\sqrt{u}} \cdot \frac{1}{2}\,du = \frac{1}{2}\int_1^4 \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\int_1^4 u^{-1/2}\,du$$

$$= \left[\frac{1}{2}\,\frac{u^{1/2}}{1/2}\right]_1^4 = \left[\sqrt{u}\right]_1^4 = \sqrt{4} - \sqrt{1} = \underline{\underline{1}}$$

$$\int_0^{3/8\pi} \sin(2x + \pi/4)\, dx$$

$u = 2x + \pi/4$; $u' = 2$; $\frac{du}{dx} = 2$; $\frac{1}{2} du = dx$

$x_1 = 0$; $u_1 = 2 \cdot 0 + \pi/4 = \pi/4$

$x_2 = \frac{3}{8}\pi$; $u_1 = 2 \cdot \frac{3}{8}\pi + \pi/4 = \pi$

$$\int_{\pi/4}^{\pi} \sin(u) \cdot \frac{1}{2} du = \frac{1}{2} \int_{\pi/4}^{\pi} \sin(u)\, du$$

$$= \frac{1}{2} \left[-\cos(u)\right]_{\pi/4}^{\pi} = -\frac{1}{2} \left[\cos(u)\right]_{\pi/4}^{\pi}$$

$$= -\frac{1}{2} \left[\cos(\pi) - \cos(\pi/4)\right]$$

$$= -\frac{1}{2} \left[-1 - \frac{1}{2}\sqrt{2}\right] = \frac{1}{2} + \frac{1}{4}\sqrt{2}$$

$$\int_0^2 (x + \sin(1-x)\, dx$$

$u = 1-x$; $u' = -1$; $\frac{du}{dx} = -1$; $-du = dx$

$x_1 = 0$; $u_1 = 1-0 = 1$

$x_2 = 2$; $u_2 = 1-2 = -1$

$$\int_0^2 x\, dx + \int_1^{-1} \sin(u)\, du$$

$$= \left[\frac{1}{2}x^2\right]_0^2 + \left[-\cos(u)\right]_1^{-1}$$

$$= \frac{1}{2} 2^2 - 0 - \cancel{\cos(-1)} + \cancel{\cos(1)} = 2$$

$$\int_{-\pi/4}^{7/4\pi} \left(2-2\cdot\sin\left(\frac{\pi}{4}-x\right)\right)dx$$

$$u=\frac{\pi}{4}-x \;;\; u'=-1 ;\; \frac{du}{dx}=-1 ;\; -du=dx$$

$$x_1=-\frac{\pi}{4} \;;\; u_1=\frac{\pi}{4}-\left(-\frac{\pi}{4}\right)=\frac{\pi}{2}$$

$$x_2=\frac{7}{4}\pi \;;\; u_2=\frac{\pi}{4}-\frac{7}{4}\pi=-\frac{3}{2}\pi$$

$$=\int_{-\pi/4}^{7/4\pi} 2\,dx-2\int_{\pi/2}^{-3/2\pi} \sin(u)-du$$

$$=\int_{-\pi/4}^{7/4\pi} 2\,dx+2\int_{\pi/2}^{-3/2\pi} \sin(u)\,du$$

$$=\left[2x\right]_{-\pi/4}^{7/4\pi}+2\left[-\cos(u)\right]_{\pi/2}^{-3/2\pi}$$

$$=2\cdot\frac{7}{4}\pi+2\cdot\frac{\pi}{4}-2\overset{0}{\cancel{\cos\left(-\frac{3}{2}\pi\right)}}+2\overset{0}{\cancel{\cos\left(\frac{\pi}{2}\right)}}$$

$$=\frac{16}{4}\pi=4\pi$$

$$\int_0^{\pi/2} \sin(x) \cdot \cos^2(x)\, dx$$

$u = \cos(x)$; $u' = -\sin(x)$; $\frac{du}{dx} = -\sin(x)$

$x_1 = 0$; $u_1 = \cos(0) = 1$ $\qquad -du = \sin(x)dx$

$x_2 = \pi/2$; $u_2 = \cos(\pi/2) = 0$

$$\int_1^0 u^2 \cdot -du = \int_0^1 u^2 du = \left[\frac{1}{3}u^3\right]_0^1 = \frac{1}{3}$$

$$\int_0^{\pi} \frac{1-\sin(x)}{\cos^2(x)}\,dx = \underbrace{\int_0^{\pi} \frac{1}{\cos^2(x)}}_{\text{1. Integ.}} - \underbrace{\int_0^{\pi} \frac{\sin(x)}{\cos^2(x)}\,dx}_{\text{2. Integr.}}$$

1. Integral (Nr. 238) aus Formelsammlg

$n=2$; $c=1$

$$= \left[\frac{\sin(x)}{\cos(x)}\right]_0^{\pi} = \tan(\pi) - \tan(0) = 0$$

2. Integral

$$\int_0^{\pi} \frac{\sin(x)}{\cos^2(x)}\,dx$$

$u = \cos(x)$; $u' = -\sin(x)$; $\frac{du}{dx} = -\sin(x)$

$-du = \sin(x)\,dx$

$x_1 = 0$; $u_1 = \cos(0) = 1$

$x_2 = \pi$; $u_2 = \cos(\pi) = -1$

$$\int_1^{-1} \frac{1}{u^2} - du = \int_{-1}^{1} u^{-2}\,du = \left[-\frac{1}{u}\right]_{-1}^{1} =$$

$$= -\frac{1}{1} + \frac{1}{1} = 0$$

$$\int_0^{\pi/4} \frac{\tan(x)}{\cos^2(x)}\,dx$$

$u = \tan(x)$; $u' = \frac{1}{\cos^2(x)}$; $\frac{du}{dx} = \frac{1}{\cos^2(x)}$

$x_1 = 0$; $u_1 = \tan(0) = 0$ $\qquad du = \frac{1}{\cos^2(x)}\,dx$

$x_2 = \pi/4$; $u_2 = \tan(\pi/4) = 1$

$$\int_0^1 u\,du = \left[\frac{1}{2}u^2\right]_0^1 = \frac{1}{2}$$

$$\int_0^6 \frac{16x}{x^2+4}\,dx$$

$u = x^2+4$; $u' = 2x$; $\frac{du}{dx} = 2x$; $du = 2x\,dx$

$x_1 = 0$; $u_1 = 0^2+4 = 4$

$x_2 = 6$; $u_2 = 6^2+4 = 40$

$$= 8\int_4^{40} \frac{2x}{u}\,dx = 8\int_4^{40} \frac{1}{u}\,du = 8\cdot\left[\ln|u|\right]_4^{40}$$

$$= 8\cdot\left[\ln|40| - \ln|4|\right] = 8\cdot\ln\left|\frac{40}{4}\right|$$

$$= 8\cdot\ln|10|$$

$$\int_0^5 \frac{2x}{\sqrt{x^2+4}}\,dx$$

$u = x^2+4 \; ; \; u' = 2x \; ; \; \frac{du}{dx} = 2x \; ; \; du = 2x\,dx$

$x_1 = 0 \; ; \; u_1 = 0^2+4 = 4$

$x_2 = 5 \; ; \; u_2 = 5^2+4 = 29$

$$= \int_4^{29} \frac{1}{\sqrt{u}}\,du = \int_4^{29} u^{-1/2}\,du = \left[\frac{u^{1/2}}{1/2} \right]_4^{29}$$

$$= \left[2 \cdot \sqrt{u} \right]_4^{29} = 2\sqrt{29} - 2\sqrt{4} = 2\sqrt{29} - 4$$

$$\int_0^5 \ln|6-x|\,dx$$

$u = 6-x$; $u' = -1$; $\frac{du}{dx} = -1$; $-du = dx$

$x_1 = 0$; $u_1 = 6-0 = 6$

$x_2 = 5$; $u_2 = 6-5 = 1$

$$= -\int_0^5 \ln|u|\,du = \int_1^6 \ln|u|\,du$$

$$= \left[u \cdot \ln|u| - u\right]_1^6 \quad \text{(Aus Formelsam.)}$$

$$= 6 \cdot \ln|6| - 6 - 1 \cdot \underbrace{\cancel{\ln|1|}}_{0} + 1$$

$$= 6 \cdot \ln|6| - 5$$

$$\int_1^3 \ln(x^2)\,dx$$

Nullstellen: $\ln(1) = 0$

$$x^2 = 1$$

$$x = (\pm)1$$

$$= 2 \cdot \int_1^3 \ln|x|\,dx$$

$$= 2 \cdot \left[x \cdot \ln|x| - x\right]_1^3$$

$$= 2 \cdot \left[3 \cdot \ln|3| - 3 - 1 \cdot \overset{0}{\cancel{\ln|1|}} + 1\right]$$

$$= 6 \cdot \ln|3| - 4$$

$f(x) = -\cos\left(\frac{x}{2} + \frac{\pi}{4}\right) dx$

Nullstellen:

$\frac{x}{2} + \frac{\pi}{4} = \pi/2 \quad | \cdot 4$

$2x + \pi = 2\pi \quad | -\pi$

$2x = \pi \quad | :2$

$x_1 = \pi/2$

$\frac{x}{2} + \frac{\pi}{4} = \frac{3}{2}\pi \quad | \cdot 4$

$2x + \pi = 6\pi \quad | -\pi$

$2x = 5\pi \quad | :2$

$x_2 = \frac{5}{2}\pi$

$$= -\int_{\pi/2}^{5/2\pi} \cos\left(\frac{x}{2} + \frac{\pi}{4}\right) dx$$

$u = \frac{x}{2} + \frac{\pi}{4} \;;\; u' = \frac{1}{2} \;;\; \frac{du}{dx} = \frac{1}{2} \;;\; 2du = dx$

$x_1 = \pi/2 \;;\; u_1 = \pi/4 + \pi/4 = \pi/2$

$x_2 = 5/2\pi \;;\; u_2 = 5/4\pi + \pi/4 = 3/2\pi$

$$= -\int_{\pi/2}^{3/2\pi} \cos(u)\, 2du = -2 \int_{\pi/2}^{3/2\pi} \cos(u)\, du$$

$$= -2 \cdot \left[\sin(u)\right]_{\pi/2}^{3/2\pi} = -2\left[\sin\left(\tfrac{3}{2}\pi\right) - \sin\left(\tfrac{\pi}{2}\right)\right]$$

$$= -2\left[-1 - 1\right] = 4$$

$f(x) = \sin(x) + \sin(2x)$

Wir brauchen die ersten beiden Nullstellen

Aus der Formelsammlung:

$\sin(2x) = 2 \cdot \sin(x) \cdot \cos(x)$

$\sin(x) + 2 \cdot \sin(x) \cdot \cos(x) = 0$

$\sin(x)(1 + 2 \cdot \cos(x)) = 0$

$x_1 = 0$; $x_2 = \pi$ $\quad$ $1 + 2 \cdot \cos(x) = 0$

$\cos(x) = -\frac{1}{2} \Rightarrow x_3 = \frac{2}{3}\pi$

$$\int_0^{2/3\pi} (\sin(x) + \sin(2x))dx = \underbrace{\int_0^{2/3\pi} \sin(x)dx}_{1.\,In.} + \underbrace{\int_0^{2/3\pi} \sin(2x)dx}_{2.\,In.}$$

1. Integral:

$$\int_0^{2/3\pi} \sin(x)dx = [-\cos(x)]_0^{2/3\pi} = [-\cos(\tfrac{2}{3}\pi) + \cos(0)] = \frac{1}{2} + 1 = \frac{3}{2}$$

2 Integral:

$$\int_0^{2/3\pi} \sin(2x)dx$$

$u = 2x$; $u' = 2$; $\frac{du}{dx} = 2$; $\frac{1}{2}du = dx$

$x_1 = 0$; $u_1 = 2 \cdot 0 = 0$

$x_2 = 2/3\pi$; $u_2 = 2 \cdot 2/3\pi = 4/3\pi$

$$\int_0^{4/3\pi} \sin(u)\frac{1}{2}du = \frac{1}{2}\int_0^{4/3\pi} \sin(u)du = \frac{1}{2}[-\cos(u)]_0^{4/3\pi}$$

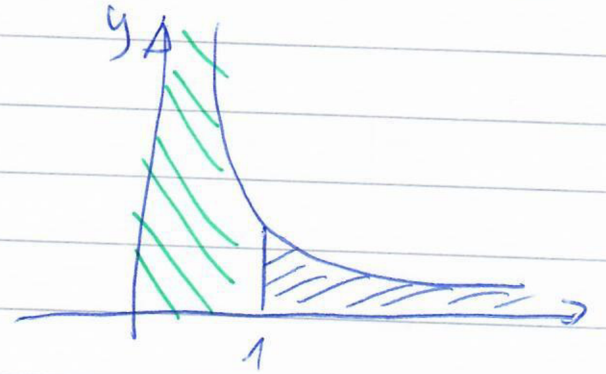
$$= \frac{1}{2} \cdot [-\cos(\tfrac{4}{3}\pi) + \cos(0)] = \frac{1}{2}[-\tfrac{1}{2} + 1] \tfrac{1}{4}$$

Gesamt: $\frac{3}{2} + \frac{1}{4} = \frac{6}{4} + \frac{1}{4} = \frac{7}{4}$

Uneigentliche Integrale

Es gibt Flächen, bei denen kaum keine Grenze angegeben werden, da die Begrenzung sich nur annähert oder sich eine Definitionslücke nähert.

Beispiel: $f(x) = 1/x^2$ im 1. Quadranten



$$\int_1^{+\infty} \frac{1}{x^2}\,dx = \lim_{a \to +\infty} \int_1^a \frac{1}{x^2}\,dx = \lim_{a\to\infty} \int_1^a x^{-2}\,dx$$

$$= \lim_{a\to\infty} \left[-\frac{1}{x}\right]_1^a = \lim_{a\to+\infty} \Big(\underbrace{-\frac{1}{a}}_{\to 0} + \frac{1}{1}\Big) = 1$$

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{b\to 0} \int_b^1 \frac{1}{x^2}\,dx = \lim_{b\to 0} \int_b^1 x^{-2}\,dx$$

$$= \lim_{b\to 0} \left[-\frac{1}{x}\right]_b^1 = \lim_{b\to 0} \Big[-\frac{1}{1} + \underbrace{\frac{1}{b}}_{\to +\infty}\Big]$$

⇒ keinen Grenzwert für diese Fläche

Lösen Sie bitte die
Aufgabe 65
aus ihre Aufgaben-
sammlung

$f(x) = (9 - x^2)\sqrt{x}$ NS: $x_1 = 0$; $x_2 = 3$

$$\int_0^3 (9-x^2)\sqrt{x}\,dx = \int_0^3 (9\sqrt{x} - x^2\cdot\sqrt{x})\,dx$$

$$= \int_0^3 (9x^{1/2} - x^{5/2})\,dx = \left[\frac{9\cdot x^{3/2}}{3/2} - \frac{x^{7/2}}{7/2}\right]_0^3$$

$$= \left[\frac{18}{3}x\cdot\sqrt{x} - \frac{2}{7}\cdot x^3\cdot\sqrt{x}\right]_0^3$$

$$= \frac{18}{3}\cdot 3\cdot\sqrt{3} - \frac{2}{7}\cdot 3^3\cdot\sqrt{3} = 18\sqrt{3} - \frac{54}{7}\sqrt{3}$$

$$= \frac{72}{7}\sqrt{3}$$

Lösen Sie folgende Aufgaben

a) $\int_{1}^{\infty} \frac{1}{x^4} \, dx$

b) $\int_{3}^{\infty} \frac{4+t}{t^3} \, dt$

c) $\int_{-\infty}^{0} e^{-t} \, dt$

d) $\int_{0}^{2} \frac{2}{x^2} \, dx$

e) $\int_{0}^{4} \frac{2}{\sqrt{t}} \, dt$

a) $\int_1^{\infty} \frac{1}{x^4}\,dx = \lim_{a\to\infty} \int_1^{a} x^{-4}\,dx$

$= \lim_{a\to\infty} \left[\frac{x^{-3}}{-3}\right]_1^a = \lim_{a\to\infty} \left[-\frac{1}{3x^3}\right]_1^a$

$= \lim_{a\to\infty} \left[-\underbrace{\cancel{\frac{1}{3\cdot a^3}}}_{0} + \frac{1}{3}\right] = \frac{1}{3}$

b) $\int_3^{\infty} \frac{4+t}{t^3}\,dt = \lim_{a\to\infty} \int_3^{a} \frac{4+t}{t^3}\,dt$

$= \lim_{a\to\infty} \int_3^{a} \left(\frac{4}{t^3} + \frac{t}{t^3}\right)dt = \lim_{a\to\infty} \int_3^{a} \left(4t^{-3} + t^{-2}\right)dt$

$= \lim_{a\to\infty} \left[-\frac{2}{t^2} - \frac{1}{t}\right]_3^a = \lim_{a\to\infty} \left[-\underbrace{\cancel{\frac{2}{a^2}}}_{0} - \underbrace{\cancel{\frac{1}{a}}}_{0} + \frac{2}{9} + \frac{1}{3}\right]$

$= 5/9$

c) $\int_{-\infty}^{0} e^{-t}\,dt = \lim_{a \to -\infty} \int_{a}^{0} e^{-t}\,dt$

$= \lim_{a \to -\infty} \left[-e^{-t}\right]_a^0 = \lim_{a \to -\infty} \left[-e^{0} + e^{-a}\right]$

$\Rightarrow$ existiert nicht

d) $\int_{0}^{2} \frac{2}{x^2}\,dx = \lim_{a \to 0} \int_{a}^{2} 2x^{-2}\,dx$

$= \lim_{a \to 0} \left[\frac{2 \cdot x^{-1}}{-1}\right]_a^2 = \lim_{a \to 0} \left[-\frac{2}{x}\right]_a^2$

$= \lim_{a \to 0} \left[-\frac{2}{2} + \frac{2}{a}\right] \Rightarrow$ existiert nicht

e) $\int_{0}^{4} \frac{2}{\sqrt{t}}\,dt = \lim_{a \to 0} \int_{a}^{4} 2 \cdot t^{-1/2}\,dt$

$\overset{\lim_{a \to 0}}{=} 2 \cdot \int_{a}^{4} t^{-1/2}\,dt \overset{\lim_{a \to 0}}{=} 2 \cdot \left[\frac{t^{1/2}}{1/2}\right]_a^4$

$\overset{\lim_{a \to 0}}{=} 4 \cdot \left[\sqrt{t}\right]_a^4 = \lim_{a \to 0} 4 \cdot \sqrt{4} - 4\underbrace{\sqrt{a}}_{0} = 8$

$f(x) = \sqrt{4-x}$ Nullstelle bei $x=4$

$$\int_0^4 \sqrt{4-x}\, dx$$

$u = 4-x$; $u' = -1$; $\frac{du}{dx} = -1$; $-du = dx$
$x_1 = 0$; $u_1 = 4-0 = 4$
$x_2 = 4$; $u_2 = 4-4 = 0$

$$= \int_4^0 \sqrt{u}\, -du = \int_0^4 u^{1/2} du = \left[\frac{u^{3/2}}{3/2} \right]_0^4$$

$$= \left[\frac{2}{3} \cdot u \cdot \sqrt{u} \right]_0^4 = \frac{2}{3} \cdot 4 \cdot \sqrt{4} = \frac{16}{3}$$

$f(x) = x \cdot \sqrt{6-x}$ Nullstellen: $x_1 = 0$; $x_2 = 6$

$$\int_0^6 (x \cdot \sqrt{6-x})\,dx$$

$u = 6-x$; $u' = -1$; $\frac{du}{dx} = -1$; $-du = dx$

$x = 6-u$

$x_1 = 0$; $u_1 = 6-0 = 6$

$x_2 = 6$; $u_2 = 6-6 = 0$

$$= \int_6^0 (6-u)\sqrt{u} \cdot -du = \int_0^6 (6u^{1/2} - u^{3/2})\,du$$

$$= \left[\frac{6 \cdot u^{3/2}}{3/2} - \frac{u^{5/2}}{5/2} \right]_0^6 = \left[4 \cdot u\sqrt{u} - \frac{2}{5}u^2\sqrt{u} \right]_0^6$$

$$= 4 \cdot 6 \cdot \sqrt{6} - \frac{2}{5} \cdot 6^2 \sqrt{6} = 24\sqrt{6} - \frac{72}{5}\sqrt{6}$$

$$= \frac{48}{5}\sqrt{6}$$

$$f(x) = \frac{2x}{\sqrt{x^2+4}}$$ Nullstelle bei $x=0$

$$\int_0^5 \frac{2x}{\sqrt{x^2+4}}\,dx$$

$u = x^2+4$; $u' = 2x$; $\frac{du}{dx} = 2x$; $du = 2x\,dx$

$x_1 = 0$; $u_1 = 0^2+4 = 4$

$x_2 = 5$; $u_2 = 5^2+4 = 29$

$$= \int_4^{29} \frac{1}{\sqrt{u}}\,du = \int_4^{29} u^{-1/2}\,du = \left[\frac{u^{1/2}}{1/2}\right]_4^{29}$$

$$= \left[2\sqrt{u}\right]_4^{29} = 2\sqrt{29} - 2\cdot\sqrt{4} = 2\sqrt{29} - 4$$